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LETTER TO THE EDITOR

Modelling tsunamis

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Abstract
We doubt the relevance of soliton theory to the modelling of tsunamis, and present a case in support of an alternative view. Although the shallow-water equations do provide, we believe, an appropriate basis for this phenomenon, an asymptotic analysis of the solution for realistic variable depths, and for suitable background flows, is essential for a complete understanding of this phenomenon. In particular we explain how a number of tsunami waves can arrive at a shoreline.

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Tsunamis have intrigued us, and frightened us, ever since the existence of the phenomena could be transmitted by word of mouth, or the statements of witnesses were published. Furthermore, the most recent disaster (December 2004) has taught us that, even in this age of almost immediate world-wide awareness, and modern scientific knowledge and skills, we are unable to predict the appearance of a tsunami and suitably prepare for its arrival. However, we are able to use appropriate fundamental equations of fluid mechanics, together with some carefully-chosen modelling, to construct a theory, and asymptotic solutions, for this phenomenon. This will enable us to highlight the essential mechanisms that exist, and to identify the factors that govern, for example, the number of wave fronts that appears at a shoreline.

How the tsunami is initiated, by an undersea earthquake resulting in—usually—a very rapid raising of the ocean, by a few metres over a large area, is fairly well understood (even if the details are a bit hazy, and prediction is virtually impossible at present). The thrust of a mathematical approach is to examine how a wave, once initiated, moves, evolves and eventually becomes such a destructive force of nature. So we aim to describe how an initial disturbance gives rise to a tsunami wave (or, as is often observed, a few waves that arrive over a period of a few minutes—even as much as 30 min, or so). Further, we can predict what will happen if a tsunami approaches a particular beach; long, gently-sloping beaches produce the worst type of tsunami wave, whereas a deep trough close inshore greatly inhibits the formation of a large wave. In addition, we will mention that an existing background state (in the form of some prescribed vorticity) modifies the wave evolution.

It is convenient to start with some data associated with the December 2004 tsunami that was triggered off the coast of Sumatra. This particular wave, it now appears, was initiated by a massive earthquake that raised the level of the ocean by a maximum of about 10 m, over a width of about 200 km, and along a 1300 km front (although, of course, the ocean was not raised uniformly over this region); the depth of the sea associated with the wave front moving across the Indian ocean is, typically, about 4.5 km. In some places there was observed just one tsunami wave, in others as many as three (and the second was the largest, by quite a significant margin, which argues against a soliton theory for this phenomenon). Data from the recent disaster have been estimated [1, 2] and this enables us to determine the two important parameters that arise in water-wave theories (and relevant background on these theories is available [3]).

These parameters are $\varepsilon = \text{amplitude}/\text{depth}$, the amplitude parameter, and the shallowness parameter $\delta = \text{depth}/\text{wavelength}$. The former parameter is associated with the *nonlinearity* of the wave, so that small ε implies a nearly-linear wave theory. On the other hand, δ measures the deviation of the pressure (in the water below the wave) away from the hydrostatic pressure distribution; this, in turn, contributes to the *dispersion* of the wave. A balance between these two effects, normally achieved after the wave has travelled a considerable distance, heralds the appearance of the *Korteweg-de Vries* (KdV) *equation* describing, approximately, the surface wave—but this is not the route we follow here [4–7]. Indeed, the observation that the largest tsunami wave is not always the first to arrive indicates that a soliton structure [8] is not likely to be appropriate; our contention is that other factors may be important.

The parameter values for the December 2004 wave, based on estimates for the initial disturbance, are $\varepsilon \approx 0.002$ and $\delta \approx 0.04$ (in fact δ^2 appears in the governing equations). The distance over which the wave will travel before the effects of nonlinearity and dispersion are both significant, and balance, is controlled by $\delta\varepsilon^{-3/2} \times$ wavelength $\approx 90\,000$ km—far in excess of the observed distances that the wave travels across an ocean. Thus our proposal is that nonlinearity alone, enhanced by the effects of decreasing depth, completely describes the essential features of the tsunami wave. This implies, therefore, that we should use the *shallow-water equations* [9], but for variable depth (which will include beaches and shorelines). At first sight, this is very much the conventional approach, but we develop some analytical results for arbitrary depth variations, both in deep water and near the shoreline. This, we submit, is far preferable to a numerical study of the equations. What we present makes clear how the shape of the initial wave, and the depth variation, produce the tsunami. In addition, in our final version of the model, we incorporate a contribution from existing (background) vorticity of the flow field. This ingredient, we can expect, will affect the overall development of the wave in deep water, and could play an important role in the wave evolution as the beach is approached if there is significant background motion in the water here.

The dispersion of the wave is not totally ignored, although its effects are negligible for the propagation and overall evolution of the wave; it will play a role near the steep wave fronts. Here, where the spatial derivatives are large, a different approximation is needed—but it applies only locally. In particular, higher derivatives, being the analogue of the third derivative in the KdV equation, are needed to smooth the wave profile.

To summarise, we take as our proposed model the equations of inviscid, but rotational fluid mechanics, together with the relevant boundary conditions for classical water waves, and allow the wave to propagate over a depth that varies on a slow horizontal scale—an important simplifying assumption, but not untypical—with some general background vorticity which varies on the same slow scale. This latter because, we may suppose, any background flow is likely to possess a scale that corresponds to the scale of the bottom topography. Within this model, we assume that the effects of dispersion are ignored, insofar as the evolution of the wave is concerned. Further, we assume that the wave of interest is already moving in one direction, having been initiated by some suitable initial profile.

The procedure that we employ, which is quite common in fluid mechanics and wave propagation, is that of *multiple scales*. This is the only realistic approach available since we wish to examine the effects of general depth variation, and of a general background flow, on the wave. As an example of the results that we can obtain, the surface wave can be represented by the asymptotic approximation that takes the form

$$\eta \sim B^{-1/4} f + \varepsilon \frac{7Y}{4} B^{-3/2} f^2 f_\xi + \sigma B^{-1/4} \left(\frac{1}{8} \int_{-X_0}^{X} B^{-1/4} (B^{-1/4} B')' \, \mathrm{d}X \right) \int_\xi^\infty f \, \mathrm{d}\xi, \qquad \varepsilon \to 0, \quad \sigma \to 0,$$

where the bottom profile is $z = -B(X)$, $X = \sigma x$ (σ is the parameter measuring the scale on which the depth varies), the undisturbed surface is $z = 0$, $\xi = t - \sigma^{-1} \int_{-X_0}^{X} B^{-1/2} \, \mathrm{d}X$, $Y = -\frac{3}{2} \varepsilon \sigma^{-1} \int_{-X_0}^{X} B^{-7/4} \, \mathrm{d}X$, $f = F(\xi - Yf)$ (F an arbitrary function). This describes a nonlinear wave that evolves as the depth changes; the critical non-uniformity in this asymptotic expansion occurs where $B = O(\varepsilon^{4/5})$, and then the governing equations depend on the single parameter $\varepsilon^{2/5}\sigma$. The solution in the neighbourhood of the shoreline involves the full, variable depth, shallow-water equations; however, if the depth varies linearly in this region—which is typical of many beaches—then an exact solution is available [10, 11] near the shore. This can be matched to the solution (given above), which is valid in deeper water. This results in a solution which is controlled by the initial profile of the wave, given in the neighbourhood of $X = -X_0$, which defines the form taken by F. This, in turn, implies that the shape of the wave approaching the beach is governed by the form of F. Thus a wave with regions—one or more—that steepen as the depth decreases will generate the appropriate number of tsunami waves at the shore, and the sizes are completely dependent on the form of F: larger could follow smaller waves. Our model permits any choice of $B(X)$, and is readily extended to allow for any (consistent) background vorticity distribution, which produces a similarly-structured solution, but with coefficients that depend on integrals over the depth of the background state.

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